

Solutions of the Cheeger problem via torsion functions ^{*}

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Abstract

The Cheeger problem for a bounded domain $\Omega \subset \mathbb{R}^N$, $N > 1$ consists in minimizing the quotients $|\partial E|/|E|$ among all smooth subdomains $E \subset \Omega$ and the Cheeger constant $h(\Omega)$ is the minimum of these quotients. Let $\phi_p \in C^{1,\alpha}(\overline{\Omega})$ be the p -torsion function, that is, the solution of torsional creep problem $-\Delta_p \phi_p = 1$ in Ω , $\phi_p = 0$ on $\partial\Omega$, where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator, $p > 1$. The paper emphasizes the connection between these problems. We prove that $\lim_{p \rightarrow 1^+} (\|\phi_p\|_{L^\infty(\Omega)})^{1-p} = h(\Omega) = \lim_{p \rightarrow 1^+} (\|\phi_p\|_{L^1(\Omega)})^{1-p}$. Moreover, we deduce the relation $\lim_{p \rightarrow 1^+} \|\phi_p\|_{L^1(\Omega)} \geq C_N \lim_{p \rightarrow 1^+} \|\phi_p\|_{L^\infty(\Omega)}$ where C_N is a constant depending only of N and $h(\Omega)$, explicitly given in the paper. An eigenfunction $u \in BV(\Omega) \cap L^\infty(\Omega)$ of the Dirichlet 1-Laplacian is obtained as the strong L^1 limit, as $p \rightarrow 1^+$, of a subsequence of the family $\{\phi_p/\|\phi_p\|_{L^1(\Omega)}\}_{p>1}$. Almost all t -level sets E_t of u are Cheeger sets and our estimates of u on the Cheeger set $|E_0|$ yield $|B_1| h(B_1)^N \leq |E_0| h(\Omega)^N$, where B_1 is the unit ball in \mathbb{R}^N . For Ω convex we obtain $u = |E_0|^{-1} \chi_{E_0}$.

1 Introduction

In this paper we consider the minimization problem

$$h(\Omega) = \min_{E \subset \Omega} \frac{|\partial E|}{|E|}, \quad (1)$$

known as the *Cheeger problem*. Here $\Omega \subset \mathbb{R}^N$ ($N > 1$) is smooth and bounded domain, the quotients $|\partial E|/|E|$ are evaluated among all smooth subdomains $E \subset \Omega$ and the quantities $|\partial E|$

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and $|E|$ denote, respectively, the $(N - 1)$ -dimensional Lebesgue perimeter of ∂E and the N -dimensional Lebesgue volume of E .

The value $h(\Omega)$ is known as the *Cheeger constant* of Ω and a corresponding minimizing subdomain E is called a *Cheeger set* of Ω .

Cheeger sets have importance in the modeling of landslides, see [11, 12], or in fracture mechanics, see [18].

On its turn, the Cheeger constant of Ω itself offers a lower bound (see [10, 20]) for the first eigenvalue $\lambda_p(\Omega)$ of the p -Laplacian operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $p > 1$, with homogeneous Dirichlet data, that is, $\lambda_p(\Omega)$ is the only positive real number that satisfies

$$\begin{cases} -\Delta_p u_p &= \lambda_p u_p^{p-1}, & \text{in } \Omega \\ u_p &= 0, & \text{on } \partial\Omega \end{cases} \quad (2)$$

for some positive function $u_p \in W_0^1(\Omega) \setminus \{0\}$.

It is well-known that

$$\lambda_p(\Omega) = \frac{\int_{\Omega} |\nabla u_p|^p dx}{\int_{\Omega} u_p^p dx} = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx} : u \in W_0^1(\Omega) \setminus \{0\} \right\}. \quad (3)$$

A strong connection between the solutions of the eigenvalue problem (2) and of the Cheeger problem (1) became evident from the remarkable work [15] by Kawohl and Fridman. In that paper they proved that

$$h(\Omega) = \lim_{p \rightarrow 1^+} \lambda_p(\Omega) \quad (4)$$

and that L^∞ -normalized family $\{u_p\}$ of positive eigenfunctions converges in L^1 (up to subsequences), as $p \rightarrow 1^+$, to a bounded function u whose level sets $E_t = \{x \in \Omega : u(x) > t\}$ are Cheeger sets for almost all $0 \leq t \leq 1$. Moreover, if Ω is convex they argued that $E_t = E_0$ for almost all $0 < t \leq 1$ and $u = c\chi_{E_0}$ (χ_A denotes the characteristic function of A). We remark that Cheeger sets are unique if Ω is convex (see [1, 5, 23]).

The function u built in [15] solves the eigenvalue problem for the 1-Laplacian $\Delta_1 = \operatorname{div}(\nabla u/|\nabla u|)$:

$$\begin{cases} -\Delta_1 &= h(\Omega), & \text{in } \Omega \\ u &= 0, & \text{on } \partial\Omega \end{cases} \quad (5)$$

formally deduced by taking $p = 1$ in (2) and keeping (4) in mind. Apparently inspired by the variational characterization of $\lambda_p(\Omega)$ in (3), Kawohl and Fridman [15] have reformulated (5) as a minimizing problem of quotients in the $BV(\Omega)$ space. Then, after verifying that $\{u_p\}$ is a bounded family in $BV(\Omega)$ and applying properties of this space, they proved the existence of a solution $u \in BV(\Omega)$ as mentioned above. Moreover, in [15] the authors clarified the equivalence between the problems (2) and (1) as well as presented some examples and properties of the Cheeger sets related to uniqueness, regularity and convexity.

A BV -formulation had already appeared in [14] for the operator Δ_1 , where some free boundary problems were introduced and interrelated through a minimization problem for a certain energy functional J_1 that generalizes, for $p = 1$, the *torsional creep* problem

$$\begin{cases} -\Delta_p \phi_p &= 1, & \text{in } \Omega \\ \phi_p &= 0, & \text{on } \partial\Omega. \end{cases} \quad (6)$$

However, the existence of Cheeger sets and the obtention of the Cheeger constant were not treated in that paper.

Since [14] and [15] the variational treatment of problems involving Δ_1 in the $BV(\Omega)$ space has been naturally adopted in the literature [1, 3, 6, 11, 12, 16]. We refer to [4] for a complete treatment of a more general Cheeger problem.

Our goal in this paper is to emphasize the strong connection between solutions of the Cheeger problem and the family $\{\phi_p\}$ of the p -torsion functions, that is, solutions of the torsional creep problem (6).

The major part of our approach connects (6) directly to (1) and some relations can be used as alternative estimates for $\lambda_p(\Omega)$ and $h(\Omega)$.

We prove that

$$\lim_{p \rightarrow 1^+} \frac{1}{\|\phi_p\|_\infty^{p-1}} = h(\Omega) = \lim_{p \rightarrow 1^+} \frac{1}{\|\phi_p\|_1^{p-1}} \quad (7)$$

where $\|\phi_p\|_\infty$ and $\|\phi_p\|_1$ denote, respectively, the L^∞ norm and the L^p norm of the p -torsion function ϕ_p .

We also deduce a Cheeger inequality involving $\|\phi_p\|_\infty$ and $\|\phi_p\|_1$:

$$|B_1| \left(\frac{h(B_1)}{h(\Omega)} \right)^N = \omega_N \left(\frac{N}{h(\Omega)} \right)^N \leq \liminf_{p \rightarrow 1^+} \frac{\|\phi_p\|_1}{\|\phi_p\|_\infty} \quad (8)$$

where $\omega_N = |B_1|$ is the volume of the unit ball $B_1 \subset \mathbb{R}^N$.

By exploring (7) and standard properties of BV -functions we obtain, as in [15] or [3, Section 2], the L^1 convergence (up to subsequences), when $p \rightarrow 1^+$, of the family $\left\{ \frac{\phi_p}{\|\phi_p\|_1} \right\}_{p \geq 1}$ for a solution $u \in BV(\Omega) \cap L^\infty(\Omega)$ of (5). In view of general properties of solutions of (5) (see [15] or [4]) the t -level sets E_t of this function are Cheeger set for almost $0 \leq t \leq \|u\|_\infty$ and, moreover, if Ω is convex, $E_t = E_0$ for almost $0 \leq t \leq \|u\|_\infty$ and $u = \frac{\chi_{E_0}}{|E_0|}$.

As consequence of the estimate (8) the function limit u satisfies

$$0 \leq u \leq \omega_N^{-1} \left(\frac{h(\Omega)}{N} \right)^N \quad \text{in } \Omega$$

implying the following estimate for the Cheeger set E_0 :

$$|B_1| h(B_1)^N \leq |E_0| h(\Omega)^N.$$

This estimate is optimal when Ω is a ball, the known case where Ω is its Cheeger set itself.

Alternatively, the same convergence result can be proved for the family $\left\{ \frac{\phi_p}{\|\phi_p\|_\infty} \right\}_{p \geq 1}$.

To obtain the characterizations of $h(\Omega)$ in (7) we explore some properties of the energy functional J_p associated to the torsional creep problem (6) and deduce an estimate relating $h(\Omega)$ and $\|\phi_p\|_1$, see equation (15). The first characterization in (7) was possible thanks to the estimate (8) that we prove inspired by the arguments of [19, Chap. 2 Sect. 5] (see also [2, Theor. 2]). However, in order to handle some limits as $t \rightarrow 1^+$ we had to develop some auxiliary estimates with explicit p -dependence.

We also provide a simpler proof of (7), if Ω is convex. For this we use Schwarz symmetrization and explore the concavity of $\phi_p^{1-\frac{1}{p}}$ (see [22]), which, taking into account the convexity of Ω , can be used to justify the well-known convexity of the unique Cheeger set.

The paper is organized as follows: in Section 2 we prove (7) and the given estimates of the Cheeger constant $h(\Omega)$. In Section 3 we consider the special case of a convex domain Ω , where an alternative proof of (7) is obtained and also some estimates of the Cheeger constant in terms of Beta and Gamma functions. Part of the final Section 4 is written for the convenience of the reader and reproduces the current variational approach in the BV space for the Cheeger problem (1) and some of the main results of this theory, following [4]. Then, we apply this approach to obtain Cheeger sets as level sets of a solution $u \in BV(\Omega) \cap L^\infty(\Omega)$ of (5) and state the estimate $|B_1|h(B_1)^N \leq |E_0|h(\Omega)^N$ for the Cheeger set E_0 . We end the paper by illustrating this estimate for a plane square.

2 Characterizations of the Cheeger constant

In this section we prove that

$$\lim_{p \rightarrow 1^+} \frac{1}{\|\phi_p\|_\infty^{p-1}} = h(\Omega) = \lim_{p \rightarrow 1^+} \frac{1}{\|\phi_p\|_1^{p-1}}$$

where ϕ_p is the p -torsion function of Ω , that is, the solution of (6).

It is easy to verify that the p -torsion function of a ball B_R of radius R with center at the origin is the radially symmetric function

$$\Phi_p(r) = \frac{p-1}{p} N^{-\frac{1}{p-1}} \left(R^{\frac{p}{p-1}} - r^{\frac{p}{p-1}} \right), \quad r = |x| \leq R. \quad (9)$$

Positivity, boundedness and $C^{1,\beta}$ -regularity follow from this expression. Hence, as consequence of the comparison principle and regularity theorems (see [7, 21, 25]) these properties are easily transferred to the p -torsion function of a general bounded domain Ω . Thus, one has $\phi_p > 0$ in Ω ,

$$\|\phi_p\|_\infty \leq \frac{p-1}{p} N^{-\frac{1}{p-1}} R^{\frac{p}{p-1}}$$

for any $R > 0$ such that $\Omega \subset B_R$ and $\phi_p \in C^{1,\beta}(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ for some $0 < \beta < 1$.

It follows from (6) that

$$\int_{\Omega} |\nabla \phi_p|^{p-2} \nabla \phi_p \cdot \nabla v \, dx = \int_{\Omega} v \, dx \quad \text{for all } v \in W_0^{1,p}(\Omega) \quad (10)$$

which yields, by taking $v = \phi_p$,

$$\int_{\Omega} |\nabla \phi_p|^p \, dx = \int_{\Omega} \phi_p \, dx. \quad (11)$$

Moreover, a standard variational argument shows that ϕ_p minimizes the strictly convex energy functional $J_p: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ given by

$$J_p(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} u dx. \quad (12)$$

Lemma 1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded, smooth domain. If $\varphi \in W_0^{1,p}(\Omega)$ is nonnegative in Ω and such that $\int_{\Omega} |\nabla \varphi| dx > 0$, then*

$$\liminf_{p \rightarrow 1^+} \|\phi_p\|_1^{p-1} \geq \frac{\int_{\Omega} \varphi dx}{\int_{\Omega} |\nabla \varphi| dx}, \quad (13)$$

where ϕ_p is the p -torsion function of Ω and $\|\cdot\|_1$ stands for the L^1 -norm.

Proof. Since ϕ_p is a minimizer of the functional energy J_p in $W_0^{1,p}(\Omega)$ it follows from (10) and (12) that for all $u \in W_0^{1,p}(\Omega)$ one has

$$\left(\frac{1}{p} - 1\right) \int_{\Omega} \phi_p dx = J_p(\phi_p) \leq \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} u dx.$$

Thus,

$$\|\phi_p\|_1 \geq \frac{1}{p-1} \left(p \int_{\Omega} u dx - \int_{\Omega} |\nabla u|^p dx \right) dx \quad \text{for all } u \in W_0^{1,p}(\Omega). \quad (14)$$

Now let $\varphi \in W_0^{1,p}(\Omega)$ be nonnegative in Ω and such that $\int_{\Omega} |\nabla \varphi| dx > 0$. For a fixed ϵ , $0 < \epsilon < 1$, let c_p be the positive constant such that

$$p \int_{\Omega} \varphi dx - c_p^{p-1} \int_{\Omega} |\nabla \varphi|^p dx = \epsilon \int_{\Omega} \varphi dx,$$

that is

$$c_p^{p-1} = (p - \epsilon) \frac{\int_{\Omega} \varphi dx}{\int_{\Omega} |\nabla \varphi|^p dx}.$$

It follows from (14) with $u = c_p \varphi$ that

$$\|\phi_p\|_1 \geq \frac{c_p}{p-1} \left(p \int_{\Omega} \varphi dx - c_p^{p-1} \int_{\Omega} |\nabla \varphi|^p dx \right) = \frac{\epsilon c_p}{p-1} \int_{\Omega} \varphi dx.$$

Therefore,

$$\begin{aligned} \liminf_{p \rightarrow 1^+} \|\phi_p\|_1^{p-1} &\geq \lim_{p \rightarrow 1^+} c_p^{p-1} \left(\frac{1}{p-1} \right)^{p-1} \left(\epsilon \int_{\Omega} \varphi dx \right)^{p-1} \\ &= \lim_{p \rightarrow 1^+} c_p^{p-1} = (1 - \epsilon) \frac{\int_{\Omega} \varphi dx}{\int_{\Omega} |\nabla \varphi| dx}. \end{aligned}$$

Making $\epsilon \rightarrow 0$, (13) follows. □

Theorem 2 Let $\Omega \subset \mathbb{R}^N$ be a bounded, smooth domain and ϕ_p its p -torsion function. Then

$$h(\Omega) \leq \left(\frac{|\Omega|}{\|\phi_p\|_1} \right)^{\frac{p-1}{p}} \quad (15)$$

and

$$\lim_{p \rightarrow 1^+} \frac{1}{\|\phi_p\|_1^{p-1}} = h(\Omega). \quad (16)$$

Proof. The estimate (15) follows from Cavalieri's principle and coarea formula applied to the p -torsion function ϕ_p . In fact, since $\phi_p \in C^{1,\beta}(\overline{\Omega})$ we have

$$\int_{\Omega} \phi_p dx = \int_0^{\|\phi_p\|_{\infty}} |A_t| dt$$

and

$$\int_{\Omega} |\nabla \phi_p| dx = \int_0^{\|\phi_p\|_{\infty}} |\partial A_t| dt$$

where

$$A_t = \{x \in \Omega : \phi_p(x) > t\}.$$

Therefore, since $h(\Omega) \leq \frac{|\partial A_t|}{|A_t|}$, we have

$$\int_{\Omega} |\nabla \phi_p| dx = \int_0^{\|\phi_p\|_{\infty}} |\partial A_t| dt \geq \int_0^{\|\phi_p\|_{\infty}} h(\Omega) |A_t| dt = h(\Omega) \int_{\Omega} \phi_p dx.$$

Thus, Hölder inequality and (11) yield that

$$\begin{aligned} h(\Omega) &\leq \frac{\int_{\Omega} |\nabla \phi_p| dx}{\int_{\Omega} \phi_p dx} \\ &\leq \frac{(\int_{\Omega} |\nabla \phi_p|^p dx)^{\frac{1}{p}} |\Omega|^{1-\frac{1}{p}}}{\int_{\Omega} \phi_p dx} = \frac{(\int_{\Omega} \phi_p dx)^{\frac{1}{p}} |\Omega|^{1-\frac{1}{p}}}{\int_{\Omega} \phi_p dx} = \left(\frac{|\Omega|}{\|\phi_p\|_1} \right)^{\frac{p-1}{p}}, \end{aligned}$$

which is (15). It follows then

$$h(\Omega) \leq \liminf_{p \rightarrow 1^+} \left(\frac{|\Omega|}{\|\phi_p\|_1} \right)^{\frac{p-1}{p}} = \liminf_{p \rightarrow 1^+} \frac{1}{\|\phi_p\|_1^{p-1}}.$$

To complete the proof, we will firstly prove that $\limsup_{p \rightarrow 1^+} \frac{1}{\|\phi_p\|_1^{p-1}}$ is a lower bound to the quotients $|\partial E|/|E|$ formed by smooth subdomains $E \subset \subset \Omega$ whose boundary ∂E does not intercept $\partial \Omega$.

Let E such a domain. We approximate the characteristic function of E by a suitable non-negative function $\varphi_{\varepsilon} \in W_0^{1,p}(\Omega)$ such that $\varphi_{\varepsilon} \equiv 1$ on E , $\varphi_{\varepsilon} \equiv 0$ outside an ε -neighborhood of E

with $|\nabla\varphi_\varepsilon| = 1/\varepsilon$ on an ε -layer outside E (φ_ε can be taken Lipschitz). Then, for each $t \in [0, \varepsilon]$, denoting by Γ_t the t -layer outside E (in a such way that $\Gamma_0 \cup E = E$), it follows from (13) that

$$\begin{aligned} \limsup_{p \rightarrow 1^+} \frac{1}{\|\phi_p\|_1^{p-1}} &\leq \frac{\int_\Omega |\nabla\varphi_\varepsilon| dx}{\int_\Omega \varphi_\varepsilon dx} \\ &= \frac{\int_0^\varepsilon \int_{\partial(\Gamma_t \cup E)} \frac{1}{\varepsilon} dS_x dt}{|E| + \int_{\Gamma_\varepsilon} \varphi_\varepsilon dx} \\ &\leq \frac{\frac{1}{\varepsilon} (\int_0^\varepsilon dt) \left(\int_{\partial(\Gamma_\varepsilon \cup E)} dS_x \right)}{|E|} = \frac{\int_{\partial(\Gamma_\varepsilon \cup E)} dS_x}{|E|}. \end{aligned}$$

Therefore, making $\varepsilon \rightarrow 0^+$, we find

$$\limsup_{p \rightarrow 1^+} \frac{1}{\|\phi_p\|_1^{p-1}} \leq \frac{|\partial E|}{|E|}.$$

Now, if E touches $\partial\Omega$, we approximate E by a sequence $\{t_n E\}$ of subdomains $t_n E_n \subset\subset \Omega$ such that $t_n \rightarrow 1^-$. Since $|t_n E_n| = t_n^N |E|$ and $|\partial(t_n E)| = t_n^{N-1} |\partial E|$ we have that

$$\limsup_{p \rightarrow 1^+} \frac{1}{\|\phi_p\|_1^{p-1}} \leq \frac{|\partial(t_n E)|}{|t_n E|} = \frac{1}{t_n} \frac{|\partial E|}{|E|}.$$

Thus, as $t_n \rightarrow 1^-$ we obtain

$$\limsup_{p \rightarrow 1^+} \frac{1}{\|\phi_p\|_1^{p-1}} \leq \frac{|\partial E|}{|E|}. \quad \square$$

Remark 3 In the proof of (16) another estimate like (15) could be obtained by applying the variational characterization (3) of $\lambda_p(\Omega)$ and the well-known lower bound for $\lambda_p(\Omega)$ in terms of the Cheeger constant $h(\Omega)$ (for $2 \neq p > 1$, see [20]):

$$\left(\frac{h(\Omega)}{p} \right)^p \leq \lambda_p(\Omega).$$

In fact, it follows from (3) that $\lambda_p(\Omega) \leq (|\Omega|/\|\phi_p\|_1)^{p-1}$ (see equation (17), in the sequel). Thus,

$$\left(\frac{h(\Omega)}{p} \right)^p \leq \left(\frac{|\Omega|}{\|\phi_p\|_1} \right)^{p-1}.$$

The chosen estimate (15) emphasizes the direct connection between the p -torsion functions and the Cheeger constant $h(\Omega)$. Moreover, it follows from the last inequality that

$$h(\Omega) \leq p \left(\frac{|\Omega|}{\|\phi_p\|_1} \right)^{\frac{p-1}{p}}$$

an estimate that is slightly worse than (15), because

$$h(\Omega) \leq \left(\frac{|\Omega|}{\|\phi_p\|_1} \right)^{\frac{p-1}{p}} < p \left(\frac{|\Omega|}{\|\phi_p\|_1} \right)^{\frac{p-1}{p}}$$

for $p > 1$.

Remark 4 *The approximation argument used at the end of the last proof shows that any Cheeger set touches $\partial\Omega$. In fact, if a Cheeger set E does not touch $\partial\Omega$ then we can take $t_\varepsilon = 1 + \varepsilon > 1$ such that $t_\varepsilon E \subset \Omega$ with $t_\varepsilon E$ touching the boundary $\partial\Omega$. But this leads to a contradiction since*

$$h(\Omega) \leq \frac{|\partial(t_\varepsilon E)|}{|t_\varepsilon E|} = \frac{1}{t_\varepsilon} \frac{|\partial E|}{|E|} = \frac{1}{t_\varepsilon} h(\Omega) < h(\Omega).$$

We recall that if u is a continuous and nonnegative function defined in $\Omega \subset \mathbb{R}^N$ then the Schwarz symmetrization u^* of u is the function defined in Ω^* that satisfies (see [13])

$$\{x \in \Omega : u(x) > t\}^* = \{x \in \Omega^* : u^*(x) > t\}$$

for all $t \geq 0$, where A^* denotes the ball with center at the origin and same Lebesgue measure as A .

Let Ω be a bounded, smooth domain in \mathbb{R}^N , $N > 1$. The following lemma is a consequence of Talenti's comparison principle [24] for the p -Laplacian, which says that if u and U are, respectively, solutions of the Dirichlet problems

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta_p U = f^* & \text{in } \Omega^* \\ U = 0 & \text{on } \partial\Omega^*, \end{cases}$$

where f^* is the Schwarz symmetrization of f , then the Schwarz symmetrization u^* of u is bounded above by U , that is,

$$u^* \leq U \quad \text{in } \Omega^*.$$

Lemma 5 *Let ϕ_p and Φ_p be the p -torsion functions of the domains Ω and Ω^* , respectively. If ϕ_p^* denotes the Schwarz symmetrization of ϕ_p then*

$$\phi_p^* \leq \Phi_p \quad \text{in } \Omega^* = B_R.$$

The next result provides localization for $\lambda_p(\Omega)$. Moreover it gives an explicit lower bound to this eigenvalue which will be fundamental to deduce a uniform (with respect to p) upper bound to the quotient $\frac{\|\phi_p\|_1}{\|\phi_p\|_\infty}$ and hence to prove that $\lim_{p \rightarrow 1^+} \|\phi_p\|_\infty^{1-p} = h(\Omega)$.

Proposition 6 *If $\Omega \subset \mathbb{R}^N$ is a bounded, smooth domain, then*

$$C_{N,p} |\Omega|^{-\frac{p}{N}} \leq \|\phi_p\|_\infty^{1-p} \leq \lambda_p(\Omega) \leq |\Omega|^{p-1} \|\phi_p\|_1^{1-p} \quad (17)$$

where $\lambda_p(\Omega)$ and ϕ_p denote, respectively, the first eigenvalue of (2) and the p -torsion function of Ω ,

$$C_{N,p} = N\omega_N^{\frac{p}{N}} \left(\frac{p}{p-1} \right)^{p-1} \quad (18)$$

and $\omega_N = |B_1|$ is the volume of the unit ball in \mathbb{R}^N .

Proof. The last inequality in (17) follows from (3) applied to the function ϕ_p . In fact, by the Hölder inequality

$$|\Omega|^{1-p} \left(\int_{\Omega} \phi_p dx \right)^p \leq |\Omega|^{1-p} \left[\left(\int_{\Omega} \phi_p^p dx \right)^{\frac{1}{p}} |\Omega|^{1-\frac{1}{p}} \right]^p = \int_{\Omega} \phi_p^p dx.$$

Thus,

$$\lambda_p(\Omega) \leq \frac{\int_{\Omega} |\nabla \phi_p|^p dx}{\int_{\Omega} \phi_p^p dx} = \frac{\int_{\Omega} \phi_p dx}{\int_{\Omega} \phi_p^p dx} \leq \frac{\int_{\Omega} \phi_p dx}{|\Omega|^{1-p} \left(\int_{\Omega} \phi_p dx \right)^p} = \left(\frac{|\Omega|}{\|\phi_p\|_1} \right)^{p-1}.$$

The second inequality in (17) is consequence of applying a comparison principle to the positive eigenfunction e_p (with $\|e_p\|_{\infty} = 1$) and ϕ_p , since both vanish on $\partial\Omega$ and

$$-\Delta_p e_p = \lambda_p(\Omega) e_p^{p-1} \leq \lambda_p(\Omega) = -\Delta_p \left(\lambda_p(\Omega)^{\frac{1}{p-1}} \phi_p \right).$$

Thus,

$$0 \leq e_p \leq \lambda_p(\Omega)^{\frac{1}{p-1}} \phi_p \quad \text{in } \Omega$$

and, taking the maximum values of these functions, one obtains $1 = \|e_p\|_{\infty} \leq \lambda_p(\Omega)^{\frac{1}{p-1}} \|\phi_p\|_{\infty}$ and hence

$$\frac{1}{\|\phi_p\|_{\infty}^{p-1}} \leq \lambda_p(\Omega). \quad (19)$$

In order to prove the first inequality in (17), let Φ_p be the p -torsion function of $\Omega^* = B_R$, where B_R is the ball with center at the origin and radius R such that $|B_R| = |\Omega|$.

According to (9) we have $\|\Phi_p\|_{\infty} = \Phi_p(0)$ and so

$$\begin{aligned} \|\Phi_p\|_{\infty}^{1-p} &= \left(\frac{p}{p-1} \right)^{p-1} \frac{N}{R^p} \\ &= \left(\frac{p}{p-1} \right)^{p-1} \frac{N\omega_N^{\frac{p}{N}}}{(\omega_N R^N)^{\frac{p}{N}}} = C_{N,p} |B_R|^{-\frac{p}{N}} = C_{N,p} |\Omega|^{-\frac{p}{N}} \end{aligned}$$

where $C_{N,p}$ is defined by (18).

It follows from Lemma 5 that

$$\phi_p^* \leq \Phi_p \quad \text{in } \Omega^*.$$

Thus,

$$C_{N,p} |\Omega|^{-\frac{p}{N}} = \|\Phi_p\|_{\infty}^{1-p} \leq \|\phi_p^*\|_{\infty}^{1-p} = \|\phi_p\|_{\infty}^{1-p},$$

since the Schwarz symmetrization preserves the sup-norm. \square

Remark 7 The following inequalities are also given in Kawohl and Fridman [15, Corollary 15]

$$N \left(\frac{\omega_N}{|\Omega|} \right)^{\frac{1}{N}} \leq h(\Omega) \quad \text{and} \quad \lim_{p \rightarrow \infty} (\lambda_p(\Omega))^{\frac{1}{p}} \geq \lim_{p \rightarrow \infty} \|\phi_p\|_{\infty}^{1-p} \geq \left(\frac{\omega_N}{|\Omega|} \right)^{\frac{1}{N}}.$$

Both follow from (17).

Corollary 8 Let $\Omega \subset \mathbb{R}^N$ be a bounded and smooth domain. Then,

$$\int_{\Omega} |u|^p dx \leq \frac{|\Omega|^{\frac{p}{N}}}{C_{N,p}} \int_{\Omega} |\nabla u|^p dx$$

for all $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, where $C_{N,p}$ is given by (18).

Proof. It follows from (17) and of the variational characterization of $\lambda_p(\Omega)$ since

$$C_{N,p} |\Omega|^{-\frac{p}{N}} \leq \lambda_p(\Omega) \leq \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx}.$$

□

Theorem 9 Let ϕ_p be the p -torsion function of the bounded domain $\Omega \subset \mathbb{R}^N$. Then,

$$\liminf_{p \rightarrow 1^+} \int_{\Omega} \frac{\phi_p}{\|\phi_p\|_{\infty}} dx \geq \omega_N \left(\frac{N}{h(\Omega)} \right)^N \quad (20)$$

and

$$\lim_{p \rightarrow 1^+} \left(\int_{\Omega} \frac{\phi_p}{\|\phi_p\|_{\infty}} dx \right)^{p-1} = 1, \quad (21)$$

from what follows

$$\lim_{p \rightarrow 1^+} \frac{1}{\|\phi_p\|_{\infty}^{p-1}} = h(\Omega). \quad (22)$$

Proof. For each $0 < k < \|\phi_p\|_{\infty}$, define

$$A_k = \{x \in \Omega : \phi_p > k\}.$$

The function

$$(\phi_p - k)^+ = \max \{\phi_p - k, 0\} = \begin{cases} \phi_p - k, & \text{if } \phi_p > k \\ 0, & \text{if } \phi_p \leq k \end{cases}$$

belongs to $W_0^{1,p}(\Omega)$ since $\phi_p \in W_0^{1,p}(\Omega)$ and $\phi_p > 0$ in Ω . Therefore, taking $v = (\phi_p - k)^+$ in (10), we obtain

$$\int_{A_k} |\nabla \phi_p|^p dx = \int_{A_k} (\phi_p - k) dx. \quad (23)$$

(Note that A_k is an open set and therefore $\nabla(\phi_p - k)^+ = \nabla \phi_p$ in A_k .)

Now, we estimate $\int_{A_k} |\nabla \phi_p|^p dx$ from below. For this we apply Hölder inequality and Corollary 8 to obtain

$$\left(\int_{A_k} (\phi_p - k) dx \right)^p \leq |A_k|^{p-1} \int_{A_k} (\phi_p - k)^p dx \leq \frac{|A_k|^{p-1} |A_k|^{\frac{p}{N}}}{C_{N,p}} \int_{A_k} |\nabla \phi_p|^p dx.$$

Thus,

$$C_{N,p} |A_k|^{-\frac{p}{N}+1-p} \left(\int_{A_k} (\phi_p - k) dx \right)^p \leq \int_{A_k} |\nabla \phi_p|^p dx,$$

what yields

$$C_{N,p} |A_k|^{-\frac{p}{N}+1-p} \left(\int_{A_k} (\phi_p - k) dx \right)^p \leq \int_{A_k} (\phi_p - k) dx.$$

Hence, we obtain

$$\left(\int_{A_k} (\phi_p - k) dx \right)^{p-1} \leq \frac{1}{C_{N,p}} |A_k|^{\frac{p+N(p-1)}{N}},$$

an inequality that can be rewritten as

$$\left(\int_{A_k} (\phi_p - k) dx \right)^{\frac{N(p-1)}{p+N(p-1)}} \leq C_{N,p}^{-\frac{N}{p+N(p-1)}} |A_k|. \quad (24)$$

Let us define

$$f(k) := \int_{A_k} (\phi_p - k) dx = \int_k^\infty |A_t| dt$$

where the last equality follows from Cavalieri's principle.

Since $f'(k) = -|A_k|$, (24) implies that

$$1 \leq -C_{N,p}^{-\frac{N}{p+N(p-1)}} f(k)^{-\frac{N(p-1)}{p+N(p-1)}} f'(k). \quad (25)$$

Therefore, since $f(k) > 0$ and

$$f(0) = \int_{\Omega} \phi_p dx$$

integration of (25) yields an upper bound of k whenever $|A_k| > 0$:

$$\begin{aligned} k &\leq \frac{p+N(p-1)}{p} C_{N,p}^{-\frac{N}{p+N(p-1)}} \left[f(0)^{\frac{p}{p+N(p-1)}} - f(k)^{\frac{p}{p+N(p-1)}} \right] \\ &\leq \frac{p+N(p-1)}{p} C_{N,p}^{-\frac{N}{p+N(p-1)}} \left(\int_{\Omega} \phi_p dx \right)^{\frac{p}{p+N(p-1)}}. \end{aligned}$$

This means that

$$\|\phi_p\|_{\infty} \leq \frac{p+N(p-1)}{p} C_{N,p}^{-\frac{N}{p+N(p-1)}} \left(\int_{\Omega} \phi_p dx \right)^{\frac{p}{p+N(p-1)}},$$

which is equivalent to

$$\int_{\Omega} \frac{\phi_p}{\|\phi_p\|_{\infty}} dx \geq C_{N,p}^{\frac{N}{p}} \left(\frac{p}{p+N(p-1)} \right)^{\frac{p+N(p-1)}{p}} \|\phi_p\|_{\infty}^{\frac{N(p-1)}{p}}. \quad (26)$$

Now, since (18) gives that

$$\lim_{p \rightarrow 1^+} C_{N,p}^{\frac{N}{p}} \left(\frac{p}{p+N(p-1)} \right)^{\frac{p+N(p-1)}{p}} = \omega_N N^N, \quad (27)$$

we obtain (20), since it follows from (17) and (16) that

$$\liminf_{p \rightarrow 1^+} \|\phi_p\|_{\infty}^{\frac{N(p-1)}{p}} \geq \lim_{p \rightarrow 1^+} \left(\frac{\|\phi_p\|_1}{|\Omega|} \right)^{\frac{N(p-1)}{p}} = h(\Omega)^{-N}$$

Making $p \rightarrow 1^+$ in (20), we obtain (21), since

$$1 = \lim_{p \rightarrow 1^+} \left[\omega_N \left(\frac{N}{h(\Omega)} \right)^N \right]^{p-1} \leq \liminf_{p \rightarrow 1^+} \left(\int_{\Omega} \frac{\phi_p}{\|\phi_p\|_{\infty}} dx \right)^{p-1} \leq \lim_{p \rightarrow 1^+} |\Omega|^{p-1} = 1.$$

At last, we obtain from (21) and (16) that

$$\begin{aligned} \lim_{p \rightarrow 1^+} \frac{1}{\|\phi_p\|_{\infty}^{p-1}} &= \lim_{p \rightarrow 1^+} \left(\int_{\Omega} \frac{\phi_p}{\|\phi_p\|_{\infty}} dx \right)^{p-1} \lim_{p \rightarrow 1^+} \left(\int_{\Omega} \phi_p dx \right)^{1-p} \\ &= \frac{1}{\|\phi_p\|_1^{p-1}} = h(\Omega) \end{aligned}$$

and we are done. \square

Example 10 We take advantage of the expression (9) to verify directly from (22) that $h(\Omega) = \frac{|\partial\Omega|}{|\Omega|}$ if $\Omega = B_R$, a ball of radius R . In fact, for this case it follows from (22) and (9) that

$$\begin{aligned} h(B_R) &= \lim_{p \rightarrow 1^+} \|\phi_p\|_{\infty}^{1-p} \\ &= \lim_{p \rightarrow 1^+} \left(\frac{p-1}{p} N^{-\frac{1}{p-1}} R^{\frac{p}{p-1}} \right)^{1-p} \\ &= N \lim_{p \rightarrow 1^+} \left(\frac{p-1}{p} \right)^{1-p} R^{-p} = \frac{N}{R} = \frac{N\omega_N R^{N-1}}{\omega_N R^N} = \frac{|\partial B_R|}{|B_R|}. \end{aligned}$$

3 Convex domains

The main purpose of this section is to present a simpler proof of (21) for the case where Ω is convex as well as to prove the estimates

$$\frac{1}{\|\phi_p\|_\infty^{p-1}} \leq \lambda_p(\Omega) \leq \frac{1}{\|\phi_p\|_\infty^{p-1} I(q, N)^{p-1}}$$

and

$$\left(\frac{|\Omega| I(q, N)}{\|\phi_p\|_1} \right)^{p-1} \leq \lambda_p(\Omega) \leq \left(\frac{|\Omega|}{\|\phi_p\|_1} \right)^{p-1}$$

where

$$I(q, N) = N \int_0^1 (1-t)^q t^{N-1} dt \quad \text{and} \quad q = \frac{p}{p-1}.$$

For this, let B_R be the ball centered at the origin with radius R and

$$I(\alpha, N) := N \int_0^1 (1-t)^\alpha t^{N-1} dt$$

for each $\alpha > 0$ and each positive integer N . We remark that

$$I(\alpha, N) = NB(\alpha-1, N) = N \frac{\Gamma(\alpha-1)\Gamma(N)}{\Gamma(\alpha-1+N)}$$

where B and Γ are the Beta and Gamma functions, respectively.

Lemma 11 *For each positive integer N and $\alpha > 0$ one has*

$$I(\alpha, N+1) = \frac{N+1}{N+\alpha+1} I(\alpha, N). \quad (28)$$

Moreover,

$$\lim_{\alpha \rightarrow \infty} I(\alpha, N)^{\frac{1}{\alpha}} = 1. \quad (29)$$

Proof. We have

$$\begin{aligned} (\alpha+1) \int_0^1 (1-t)^\alpha t^N dt &= [-(1-t)^{\alpha+1} t^N]_0^1 + \int_0^1 (1-t)^{\alpha+1} N t^{N-1} dt \\ &= N \int_0^1 (1-t)^{\alpha+1} t^{N-1} dt \\ &= N \int_0^1 (1-t)^\alpha t^{N-1} dt - N \int_0^1 (1-t)^\alpha t^N dt \\ &= I(\alpha, N) - N \int_0^1 (1-t)^\alpha t^N dt, \end{aligned}$$

thus proving (28), since

$$\frac{I(\alpha, N+1)}{N+1} = \int_0^1 (1-t)^\alpha t^N dt = \frac{1}{N+\alpha+1} I(\alpha, N).$$

The proof of (29) follows by induction. In fact,

$$\lim_{\alpha \rightarrow \infty} I(\alpha, 1)^{\frac{1}{\alpha}} = \lim_{\alpha \rightarrow \infty} \left(\int_0^1 (1-t)^\alpha dr \right)^{\frac{1}{\alpha}} = \lim_{\alpha \rightarrow \infty} \left(\frac{1}{\alpha+1} \right)^{\frac{1}{\alpha}} = 1$$

and by assuming that $\lim_{\alpha \rightarrow \infty} I(\alpha, N)^{\frac{1}{\alpha}} = 1$ we obtain from (28) that

$$\lim_{\alpha \rightarrow \infty} I(\alpha, N+1)^{\frac{1}{\alpha}} = \lim_{\alpha \rightarrow \infty} \left(\frac{N+1}{N+\alpha+1} \right)^{\frac{1}{\alpha}} \lim_{\alpha \rightarrow \infty} I(\alpha, N)^{\frac{1}{\alpha}} = 1. \quad \square$$

Lemma 12 *If $\alpha > 0$, then*

$$\frac{1}{|B_R|} \int_{B_R} \left(1 - \frac{|x|}{R} \right)^\alpha dx = I(\alpha, N).$$

Proof. Let $\omega_N = |B_1|$. We have

$$\begin{aligned} \frac{1}{|B_R|} \int_{B_R} \left(1 - \frac{|x|}{R} \right)^\alpha dx &= \frac{1}{R^N \omega_N} \int_{B_1} (1-|y|)^\alpha R^N dy \\ &= \frac{1}{\omega_N} \int_0^1 \int_{\partial B_r} (1-|y|)^\alpha dS_y dr \\ &= \frac{1}{\omega_N} \int_0^1 (1-r)^\alpha \int_{\partial B_r} dS_x dr \\ &= N \int_0^1 (1-r)^\alpha r^{N-1} dr = I(\alpha, N). \end{aligned} \quad \square$$

Theorem 13 *Suppose that Ω is convex. Then,*

$$I(q, N) \leq \frac{1}{|\Omega|} \int_{\Omega} \frac{\phi_p}{\|\phi_p\|_\infty} dx \leq 1, \quad (30)$$

where $q = \frac{p}{p-1}$, producing a simpler proof of (21). Moreover, we have

$$\frac{1}{\|\phi_p\|_\infty^{p-1}} \leq \lambda_p(\Omega) \leq \frac{1}{\|\phi_p\|_\infty^{p-1} I(q, N)^{p-1}}, \quad (31)$$

$$\left(\frac{|\Omega| I(q, N)}{\|\phi_p\|_1} \right)^{p-1} \leq \lambda_p(\Omega) \leq \left(\frac{|\Omega|}{\|\phi_p\|_1} \right)^{p-1} \quad (32)$$

and also

$$\lim_{p \rightarrow 1^+} \frac{1}{\|\phi_p\|_\infty^{p-1}} = h(\Omega) = \lim_{p \rightarrow 1^+} \frac{1}{\|\phi_p\|_\infty^{p-1}}. \quad (33)$$

Proof. The second inequality in (30) is obvious since $\phi_p \leq \|\phi_p\|_\infty$ in Ω .

For each $p > 1$, take x_p such that $\phi_p(x_p) = \|\phi_p\|_\infty$ and consider the function $\Psi_p \in C(\overline{\Omega})$ whose graph in $\mathbb{R}^N \times \mathbb{R}$ is the cone of basis Ω and height 1 reached at x_p (thus, $\Psi_p = 0$ on $\partial\Omega$ and $\|\Psi_p\|_\infty = \Psi_p(x_p) = 1$).

Since Ω is convex, it follows from [22, Thm 2] that $\phi_p^{\frac{1}{q}}$ is concave. So, we have

$$\left(\frac{\phi_p}{\|\phi_p\|_\infty} \right)^{\frac{1}{q}} \geq \Psi_p \text{ in } \Omega.$$

Now, let $R > 0$ be such that $|B_R| = |\Omega|$ and let ϕ_p^* and Ψ_p^* denote the Schwarz symmetrizations of ϕ_p and Ψ_p , respectively. Thus, both ϕ_p^* and Ψ_p^* are positive and radially symmetric decreasing in B_R .

Moreover,

- $\phi_p^* = 0 = \Psi_p^*$ on ∂B_R ;
- $\|\phi_p\|_\infty = \|\phi_p^*\|_\infty = \phi_p^*(0)$;
- $\|\Psi_p\|_\infty = \|\Psi_p^*\|_\infty = 1$;
- $\left(\phi_p^{\frac{1}{q}} \right)^* = (\phi_p^*)^{\frac{1}{q}}$, $(\Psi_p^q)^* = (\Psi_p^*)^q$;
- $\int_\Omega \phi_p dx = \int_{B_R} \phi_p^* dx$ and $\int_\Omega \Psi_p dx = \int_{B_R} \Psi_p^* dx$.

From the definition of the Schwarz symmetrization follows that

$$\Psi_p^*(x) = \left(1 - \frac{|x|}{R} \right), \quad |x| \leq R.$$

Since Schwarz symmetrization preserves order and positive powers, we also have that

$$\frac{\phi_p^*(x)}{\|\phi_p^*\|_\infty} \geq (\Psi_p^*(x))^q = \left(1 - \frac{|x|}{R} \right)^q, \quad |x| \leq R.$$

Thus, (30) is consequence of Lemma 12, since

$$\begin{aligned} \frac{1}{|\Omega|} \int_\Omega \frac{\phi_p}{\|\phi_p\|_\infty} dx &= \frac{1}{|B_R|} \int_{B_R} \frac{\phi_p^*}{\|\phi_p^*\|_\infty} dx \\ &\geq \frac{1}{|B_R|} \int_{B_R} \left(1 - \frac{|x|}{R} \right)^q dx = I(q, N). \end{aligned}$$

From (30) and (29) we obtain

$$\begin{aligned}
1 &\geq \lim_{p \rightarrow 1^+} \left(\int_{\Omega} \frac{\phi_p}{\|\phi_p\|_{\infty}} dx \right)^{p-1} \\
&\geq \lim_{p \rightarrow 1^+} |\Omega|^{p-1} \lim_{p \rightarrow 1^+} I(q, N)^{p-1} \\
&= \lim_{p \rightarrow 1^+} I(q, N)^{\frac{2}{q}} = \lim_{p \rightarrow 1^+} \left(\lim_{q \rightarrow \infty} I(q, N)^{\frac{1}{q}} \right)^p = 1
\end{aligned}$$

thus proving (21). From the last estimate and (17) we obtain (31), (32) and (33). \square

4 Cheeger sets

In this section we reproduce the current variational approach in the BV space for the Cheeger problem (1) and apply it to verify that the L^1 -normalized family $\left\{ \frac{\phi_p}{\|\phi_p\|_1} \right\}_{p \geq 1}$ converges (up to subsequences) in $L^1(\Omega)$, when $p \rightarrow 1^+$, to a function $u \in L^1(\Omega) \cap L^{\infty}(\Omega)$ whose t -level sets E_t are Cheeger sets. Moreover, under convexity of Ω we verify that $u = |E_0|^{-1} \chi_{E_0}$ where χ_{E_0} denotes the characteristic function of the Cheeger set E_0 . The function u also solves the problem

$$\begin{cases} -\Delta_1 &= h(\Omega), & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{cases} \quad (34)$$

in a sense to be clarified in the sequence (Remark 18).

For each $v \in L^1(\Omega)$, let $\int_{\Omega} |Dv| dx$ denote the *variation* of v in Ω which is defined by

$$\int_{\Omega} |Dv| dx = \sup \left\{ \int_{\Omega} v \operatorname{div} g : g \in C_0^1(\Omega, \mathbb{R}^N) \text{ and } \|g\|_{\infty} \leq 1 \right\}.$$

Note that $\int_{\Omega} |Dv| dx$ is defined in terms of the weak (distributional) derivative of u . Moreover, the variation of a function $v \in C^1(\Omega)$ coincides with the L^1 -norm of its gradient, that is

$$\int_{\Omega} |Dv| dx = \int_{\Omega} |\nabla v| dx \quad \text{when } v \in C^1(\Omega).$$

The space $BV(\Omega)$ of the *bounded variation functions* is then defined by

$$BV(\Omega) = \left\{ v \in L^1(\Omega) : \int_{\Omega} |Dv| dx < \infty \right\}.$$

It is known (see [8], [9]) that $BV(\Omega)$ is a Banach space with the norm

$$\|v\|_{BV} := \int_{\Omega} |v| dx + \int_{\Omega} |Dv| dx$$

and, moreover, the following properties hold (see [8, Section 5.2]):

Lemma 14 (lower semicontinuity) *If $v_n \rightarrow v$ in $L^1(\Omega)$ then*

$$\int_{\Omega} |Dv| dx \leq \liminf_n \int_{\Omega} |Dv_n| dx.$$

Lemma 15 (L^1 -compactness) *If $\{v_n\}_{n \in \mathbb{N}} \subset BV(\Omega)$ is a bounded sequence in the BV -norm, then (up to a subsequence) $v_n \rightarrow v$ in $L^1(\Omega)$.*

Lemma 16 (coarea formula) *Let $v \in BV(\Omega)$. Then*

$$\int_{\Omega} |Dv| dx = \int_{-\infty}^{\infty} |\partial E_t| dt.$$

(Here $E_t := \{x \in \Omega : v(x) > t\}$ is the t -level set of v and $|\partial E_t|$ denotes its perimeter in Ω .)

It is also known that when $\partial\Omega$ is Lipschitz, functions in $BV(\Omega)$ have a trace on $\partial\Omega$. Thus, from now on we assume that $\partial\Omega$ is Lipschitz.

Since a Cheeger set $E \subset \overline{\Omega}$ touches $\partial\Omega$ it is important to consider the boundary $\partial\Omega$ in the variational formulation of the Cheeger problem.

We consider the minimizing problem

$$\mu = \inf_{v \in \Lambda} H(v) \tag{35}$$

where

$$H(v) := \int_{\Omega} |Dv| dx + \int_{\partial\Omega} |v| d\mathcal{H}^{N-1} \tag{36}$$

and

$$\Lambda = \{v \in BV(\mathbb{R}^N) : v \geq 0 \text{ in } \overline{\Omega}, \ v \equiv 0 \text{ in } \mathbb{R}^N \setminus \overline{\Omega}, \ \|v\|_1 = 1\}.$$

In the surface integral in (36), $|v|$ denotes the internal trace of v and $d\mathcal{H}^{N-1}$ denotes the $(N-1)$ -dimensional Hausdorff measure.

We also remark (see [8]) that $H(\chi_E)$ is the perimeter of E in \mathbb{R}^N for $E \subset \overline{\Omega}$ and that if $v \in \Lambda$ then $v \in BV(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |Dv| dx = \int_{\Omega} |Dv| dx + \int_{\partial\Omega} |v| d\mathcal{H}^{N-1}.$$

Proposition 17 *It holds $\mu = h(\Omega)$.*

Proof. For an arbitrary $E \subset \overline{\Omega}$ we have

$$\frac{|\partial E|}{|E|} = \frac{H(\chi_E)}{|E|} = H\left(\frac{\chi_E}{|E|}\right) \geq \mu$$

what implies, in view of (1), that $\mu \leq h(\Omega)$. On the other hand, if $v \in \Lambda$ it follows from Lemma 16 and Cavalieri's principle that

$$\begin{aligned} H(v) &= \int_{\mathbb{R}^N} |Dv| \, dx \\ &= \int_0^\infty |\partial E_t| \, dt \\ &= \int_0^\infty \frac{|\partial E_t|}{|E_t|} |E_t| \, dt \\ &\geq h(\Omega) \int_0^\infty |E_t| \, dt = h(\Omega) \|v\|_1 = h(\Omega). \end{aligned}$$

Since $v \in \Lambda$ is arbitrary, we conclude from (35) that $h(\Omega) \leq \mu$. \square

Remark 18 *Since $\mu = h(\Omega)$, the problem (35) can be considered as a variational formulation of (34). In view (4) such a solution is considered as an eigenvalue of (34). For details we refer to [15, Remark 7].*

The existence of a Cheeger set $E \subset \overline{\Omega}$ is equivalent to finding a minimizer u for the problem (35) in the following sense:

Proposition 19 *If u minimizes (35), then its t -level sets*

$$E_t := \{x \in \Omega : u(x) > t\}$$

satisfying $|E_t| > 0$ are Cheeger sets. In particular, E_0 is a Cheeger set.

On the other hand, if $E \subset \overline{\Omega}$ is a Cheeger set, then $\frac{\chi_E}{|E|}$ minimizes (35).

Proof (sketch). For the first claim we present only a sketch and refer to [4, Theor. 2] for details.

Let $u \in \Lambda$ be a minimizer of (35) and define

$$T_n(v) = \begin{cases} 0 & \text{if } 0 < v \\ nv & \text{if } 0 \leq v < \frac{1}{n} \\ 1 & \text{if } v \geq \frac{1}{n}. \end{cases}$$

For n large enough the function $w_n = \frac{T_n(u)}{\|T_n(u)\|_1}$ also minimizes (35) in Λ . Hence the convergence in L^1 of w_n to $w_0 := \frac{\chi_{E_0}}{|E_0|} \in \Lambda$ yields that w_0 solves (35). Therefore,

$$h(\Omega) = H(w_0) = \frac{1}{|E_0|} H(\chi_{E_0}) = \frac{|\partial E_0|}{|E_0|}$$

proving that E_0 is a Cheeger set.

If $t > 0$ is such that $|E_t| > 0$ then it is possible to verify that the function $v := \frac{(u-t)_+}{\|(u-t)_+\|_1}$ solves (35). Thus, by applying the previous argument for E_0^v , the zero-level set of v , we conclude that $\frac{\chi_{E_0^v}}{|E_0^v|}$ also solves (35). Since

$$\frac{\chi_{E_0^v}}{|E_0^v|} = \frac{\chi_{E_t}}{|E_t|}$$

we are done.

Now, in order to prove the second claim, let E be a Cheeger set and take $u = \frac{\chi_E}{|E|}$. Then, $u \in \Lambda$ and

$$h(\Omega) = \frac{|\partial E|}{|E|} = \frac{H(\chi_E)}{|E|} = H\left(\frac{\chi_E}{|E|}\right) = H(u). \quad \square$$

Now we prove our main result on Cheeger sets and the minimization of H .

Theorem 20 *Let $u_p := \frac{\phi_p}{\|\phi_p\|_1}$. Then there exists a sequence $\{u_{p_n}\} \subset C_0^1(\overline{\Omega}) \cap BV(\Omega)$ and a function $u \in \Lambda \cap L^\infty(\Omega)$, such that $p_n \rightarrow 1^+$ and $u_{p_n} \rightarrow u$ in $L^1(\Omega)$. Moreover:*

- (i) $u = 0$ on $\partial\Omega$;
- (ii) $0 \leq u \leq \omega_N^{-1} \left(\frac{h(\Omega)}{N}\right)^N$ in Ω ;
- (iii) $h(\Omega) = H(u)$, that is, u minimizes (35);
- (iv) Almost all t -level sets of u are Cheeger sets for $0 \leq t \leq \|u\|_\infty$.

Proof. Since

$$\int_{\Omega} |\nabla \phi_p|^p dx = \int_{\Omega} \phi_p dx$$

we have that

$$\int_{\Omega} |\nabla u_p|^p dx = \frac{1}{\|\phi_p\|_1^{p-1}} \int_{\Omega} u_p dx = \frac{1}{\|\phi_p\|_1^{p-1}}.$$

Thus, it follows from Hölder inequality that

$$\int_{\Omega} |\nabla u_p| dx \leq \left(\int_{\Omega} |\nabla u_p|^p dx \right)^{\frac{1}{p}} |\Omega|^{1-\frac{1}{p}} = \left(\frac{1}{\|\phi_p\|_1^{p-1}} \right)^{\frac{1}{p}} |\Omega|^{1-\frac{1}{p}}.$$

Hence, since $u_p \in C^{1,\beta}(\overline{\Omega}) \cap W_0^{1,p}(\Omega) \subset C_0^1(\overline{\Omega})$ (here β may depend on p) and $\|u_p\|_1 = 1$, we have

$$\begin{aligned} \|u_p\|_{BV} &= \int_{\Omega} u_p dx + \int_{\Omega} |Du_p| dx \\ &= 1 + \int_{\Omega} |\nabla u_p| dx \\ &\leq 1 + \left(\frac{1}{\|\phi_p\|_1^{p-1}} \right)^{\frac{1}{p}} |\Omega|^{1-\frac{1}{p}} \xrightarrow[p \rightarrow 1^+]{} 1 + h(\Omega) < \infty. \end{aligned}$$

Therefore the family $\{u_p\}_{p \geq 1}$ is bounded in $BV(\Omega)$ for all p sufficiently close to 1^+ . Thus, it follows from Lemma 15 that there exists a sequence $p_n \rightarrow 1^+$ such that

$$u_{p_n} \rightarrow u \text{ in } L^1(\Omega).$$

Moreover, $\|u\|_1 = 1$ and, up to a subsequence, we can assume that $u_{p_n} \rightarrow u$ a.e. in Ω and that u satisfies properties (i) and (ii), the upper bound in (ii) being a consequence of (20).

Lemma 14 applied to the sequence $\{u_{p_n}\}$ yields

$$\begin{aligned} \int_{\Omega} |Du| dx &\leq \liminf_n \int_{\Omega} |Du_{p_n}| dx \\ &= \liminf_n \int_{\Omega} |\nabla u_{p_n}| dx \\ &\leq \lim_n \left(\int_{\Omega} |\nabla u_{p_n}|^{p_n} dx \right)^{\frac{1}{p_n}} |\Omega|^{1 - \frac{1}{p_n}} \\ &= \lim_n \left(\frac{1}{\|\phi_p\|_1^{p_n}} \int_{\Omega} |\nabla \phi_{p_n}|^p dx \right)^{\frac{1}{p_n}} \\ &= \lim_n \left(\frac{1}{\|\phi_p\|_1^{p_n}} \int_{\Omega} \phi_{p_n} dx \right)^{\frac{1}{p_n}} = \lim_n \left(\frac{1}{\|\phi_p\|_1^{p_n-1}} \right)^{\frac{1}{p_n}} = h(\Omega). \end{aligned}$$

Thus, $u \in \Lambda$ and, since $u = 0$ on $\partial\Omega$, we have

$$\begin{aligned} H(u) &= \int_{\Omega} |Du| dx + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1} \\ &= \int_{\Omega} |Du| dx \leq h(\Omega) = \inf_{v \in \Lambda} H(v) \leq H(u). \end{aligned}$$

Hence, $H(u) = h(\Omega)$, that is, u is a minimizer of (35), proving (iii).

The claim (iv) is consequence of (iii) and Proposition 19. □

Remark 21 *If Ω is convex, then the function u of the last theorem can be written as*

$$u = \|u\|_{\infty} \chi_{E_0} = \frac{\chi_{E_0}}{|E_0|}$$

where $E_0 = \{x \in \overline{\Omega} : u(x) > 0\}$. In fact, this follows from the uniqueness of the Cheeger set, since $E_0 = E_t$ for almost all t -level set E_t of u , with $0 \leq t \leq \|u\|_{\infty}$. Thus, since $\|u\|_{\infty} \chi_{E_0} \geq u$ in E_0 we have

$$\begin{aligned} \|\|u\|_{\infty} \chi_{E_0} - u\|_1 &= \int_{\Omega} |\|u\|_{\infty} \chi_{E_0} - u| dx \\ &= \int_{E_0} (\|u\|_{\infty} \chi_{E_0} - u) dx \\ &= \|u\|_{\infty} |E_0| - \int_0^{\|u\|_{\infty}} |E_t| dt \\ &= \|u\|_{\infty} |E_0| - \int_0^{\|u\|_{\infty}} |E_0| dt = 0. \end{aligned}$$

Since $\|u\|_1 = 1$ we also have $1 = \|u\|_\infty \|\chi_{E_0}\|_1 = \|u\|_\infty |E_0|$ implying that $\|u\|_\infty = \frac{1}{|E_0|}$.

Since $\frac{\chi_{E_0}}{|E_0|}$ is a Cheeger set for $E_0 = \{x \in \overline{\Omega} : u(x) > 0\}$, it is interesting to notice that the claim (ii) gives a lower bound for the volume $|E_0|$ in terms of the Cheeger constant. In fact,

$$1 = \|u\|_1 = \int_{E_0} u \, dx \leq |E_0| \|u\|_\infty \leq |E_0| \omega_N^{-1} \left(\frac{h(\Omega)}{N} \right)^N$$

implies that

$$\omega_N \left(\frac{N}{h(\Omega)} \right)^N \leq |E_0|$$

or, what is the same,

$$|B_1| h(B_1)^N \leq |E_0| h(\Omega)^N \quad (37)$$

since $\omega_N = |B_1|$ and $h(B_1) = N$.

Moreover, since $h(\Omega)|E_0| = |\partial E_0|$, we also have

$$h(\Omega)|B_1| \left(\frac{N}{h(\Omega)} \right)^N \leq |\partial E_0|.$$

Example 22 As pointed out in [15], if $\Omega = [-1, 1] \times [-1, 1]$ is the square, then

$$h(\Omega) = \frac{4 - \pi}{4 - 2\sqrt{\pi}},$$

and the (unique) Cheeger set E satisfies

$$|E| = 4 - \frac{(4 - 2\sqrt{\pi})^2}{4 - \pi} \approx 3.7587$$

and

$$|\partial E| = 8 - \frac{(8 - 2\pi)(4 - 2\sqrt{\pi})}{4 - \pi} \approx 7.0898$$

Thus, we can evaluate (37):

$$\omega_2 \left(\frac{2}{h(\Omega)} \right)^2 = 4\pi \left(\frac{4 - 2\sqrt{\pi}}{4 - \pi} \right)^2 \approx 3.532 < 3.7587 \approx |E|$$

and

$$h(\Omega)\omega_2 \left(\frac{2}{h(\Omega)} \right)^2 = \frac{4\pi}{h(\Omega)} = \frac{4\pi(4 - 2\sqrt{\pi})}{4 - \pi} \approx 6.6622 < 7.0898 \approx |\partial E|.$$

Remark 23 We remark that (37) is optimal if $\Omega = B_R$ is a ball since $E = B_R$ is the only Cheeger set and

$$|B_R|h(B_R)^N = \omega_N R^N \left(\frac{N}{R}\right)^N = \omega_N N = |B_1|h(B_1).$$

Remark 24 Taking into account Theorem 9, the L^∞ -normalization of ϕ_p also produces, when $p \rightarrow 1^+$, a function $u \in BV(\Omega) \cap L^\infty(\Omega)$ such that $\|u\|_\infty \leq 1$,

$$\omega_N \left(\frac{N}{h(\Omega)}\right)^N \leq \|u\|_1 \leq |\Omega|$$

and whose t -level sets are Cheeger sets almost all $0 \leq t \leq 1$.

Moreover, u satisfies

$$h(\Omega) = \frac{H(u)}{\int_\Omega u \, dx} \leq \frac{H(v)}{\int_\Omega v \, dx} \text{ for all } v \in BV(\Omega) \text{ satisfying } 0 \leq v \leq 1 \text{ in } \overline{\Omega}.$$

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